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LETTER TO THE EDITOR

Mixing rates and exterior forms in chaotic systems

F Christiansen†, S Isola‡, G Paladin§†, H H Rugh†

† Niels Bohr Institutet and Nordita, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark
‡ Dipartimento di Matematica e Fisica, Universitá di Camerino, and Dipartimento di Fisica, Universitá di Firenze, L.go E. Fermi 2, I-50125 Firenze, Italy
§ Dipartimento di Fisica, Università de L'Aquila, Via Vetoio, I-67010 Coppito L'Aquila, Italy

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Abstract. We propose a numerical method for extracting the spectrum of decay rates of time correlations in chaotic dynamical systems. The sum of the first p decay rates is related to the asymptotic behaviour of the time correlation of suitable exterior forms of order p. The method is applied to maps of the interval and to the Hénon map.

Time correlation functions have attracted a renewal of interest in the study of chaotic systems both from a physical and a mathematical point of view. As a matter of fact, rigorous results are obtained only for certain classes of chaotic maps, in particular those satisfying Smale's axiom A, for which an exponential decay for functions which are continuously differentiable has been proved (Ruelle 1976a). Other examples are provided by billiards (see Bunimovich 1985). In more general cases, the leading mixing rate can be extracted from a direct analysis of the large time behaviour (i.e. $\tau\to\infty$) of the correlation function $C_A(\tau)$, defined by

$$\begin{aligned} C_A(\tau) &= \lim_{T\to\infty}\frac{1}{T}\int_0^T A(x(t+\tau))A(x(t))\,\mathrm{d}t-\langle A\rangle^2 \\ &= \langle A(x(t+\tau))A(x(t))\rangle-\langle A\rangle^2 \end{aligned} \tag{1}$$

where $x(t)=f^t x(0)$ is the time evolution in the phase space R^D and A is a smooth real function on R^D (an *observable*). In the above expression, the time average is assumed to define a unique ergodic probability measure μ, the *physical measure* (Eckmann and Ruelle 1985), over the invariant set of f.

However, apart from exceptional cases, the detailed analysis of time correlations is in general a very difficult task. The essential fact (to be discussed here) is that the time behaviour of such functions is typically determined by a superposition of complex exponentials. More precisely, a spectrum of mixing rates exists and one expects

$$C_A(\tau)=\sum_k c_k e^{-\alpha_k\tau}\,\mathrm{e}^{\mathrm{i}\omega_k\tau} \tag{2}$$

where the coefficients c_k depend on the choice of the observable, whereas the *exponents* $\alpha_k-\mathrm{i}\omega_k$ do not. From (2) it follows that the power spectrum of the signal $A(x(t))$, i.e. the Fourier transform of $C_A(\tau)$, has poles located at $z_k=\omega_k+\mathrm{i}\alpha_k$. Such poles at complex frequencies are then interpreted as *resonances* of the dynamical system (Ruelle 1986). The determination of the resonances is an important characterization of the dynamics and some methods have been recently proposed (Isola 1988), (Baladi *et al* 1989), (Christiansen *et al* 1990).

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In this paper we shall use the fact that suitable combinations (e.g. sums) of the first p exponents are related to the leading decay rate of the time correlations computed for exterior forms of order p, as pointed out in the context of the theory of Fredholm determinants for axiom A systems (Ruelle 1976b). This permits us to obtain the leading mixing rates from numerical computations with appropriate p-forms. We have applied this method both to piecewise linear expanding maps of the interval, where exact results are known, and to more general non-hyperbolic systems, like the logistic map of the interval and the Hénon map of the plane.

Let us first see, in a somewhat heuristic manner, how the exponents defined above can be related to the *eigenvalues* of a transfer operator $\mathscr{L}$. For the sake of simplicity, we consider the case of one-dimensional maps $f:[0,1]\to[0,1]$, with an absolutely continuous invariant measure $\mu(\mathrm{d}x)=\psi_0(x)\,\mathrm{d}x$. In this case, ψ_0 is an eigenfunction, with eigenvalue $\nu_0=1$, of the operator $\mathscr{L}$ which maps a function h onto

$$(\mathscr{L}h)(x)=\int \delta(x-fy)h(y)\,\mathrm{d}y=\sum_{\substack{y\\ fy=x}}\frac{1}{|D_yf|}h(y) \tag{3}$$

where D_yf is the derivative of f taken at y. It is easy to realize that $\mathscr{L}$ is dual to f in the sense that

$$\int h_1(f^\tau x)h_2(x)\,\mathrm{d}x=\int h_1(x)(\mathscr{L}^\tau h_2)(x)\,\mathrm{d}x. \tag{4}$$

By virtue of this relation the spectral properties of $\mathscr{L}$ are seen to be crucial for the analysis of correlation functions. It turns out that the spectrum of this operator can be decomposed into a discrete part (i.e. the set of all the isolated eigenvalues with finite multiplicity) and an essential part, whose relative sizes depend on a delicate way on the functional space $\mathscr{L}$ is acting on (see Keller 1989, Eckmann 1989, Ruelle 1990, Collet and Isola 1990). However, in order to make the following argument clearer, we shall make the simplified assumption (which is known to be fulfilled if the map is analytic and expanding) that the integral kernel $G(x,y)=\delta(x-fy)$ admits the spectral representation

$$G(x,y)=\sum_i \nu_i\psi_i(x)\phi_i(y) \tag{5}$$

involving the eigenvalues ordered as $1=\nu_0\geq|\nu_1|\geq|\nu_2|\ldots$, as well as right eigenfunctions

$$\int G(x,y)\psi_i(y)\,\mathrm{d}y=\nu_i\psi_i(x) \tag{6}$$

and left eigenfunctions

$$\int \phi_i(x)G(x,y)\,\mathrm{d}x=\nu_i\phi_i(y) \tag{7}$$

which satisfy the orthonormality and completeness relations: $\int \phi_i(x)\psi_j(x)\,\mathrm{d}x=\delta_{ij}$ and $\Sigma_i\,\psi_i(x)\phi_i(y)=\delta(x-y)$. In particular, since ψ_0 is the invariant probability density, ϕ_0 is equal to 1. It is easy to verify that ergodicity requires

$$\int \phi_i(x)\mu(\mathrm{d}x)=\delta_{i0} \qquad |\nu_i|\leq 1 \qquad \text{and} \qquad \nu_i\neq 1 \text{ for any } i\neq 0.$$

Furthermore, the (stronger) mixing property can be expressed by the fact that $|\nu_i|<1$ for $i \neq 0$.

Now, since we have assumed the existence of an ergodic invariant measure μ, we can write

$$\langle A(x(t+\tau))A(x(t))\rangle = \int \mu(\mathrm{d}x)A(f^\tau x)A(x). \tag{8}$$

Thereafter, using the definition (3) of the transfer operator as well as property (4), we get

$$\begin{aligned}\langle A(x(t+\tau))A(x(t))\rangle &= \int\int \mathrm{d}y\, \mu(\mathrm{d}x)A(y)\delta(y-f^\tau x)A(x)\\ &= \int \mathrm{d}y\, A(y)(\mathscr{L}^\tau A\psi_0)(y)\end{aligned} \tag{9}$$

where we have used $\mu(\mathrm{d}x)=\psi_0(x)\,\mathrm{d}x$. This is a general expression which is valid regardless of the choice of the observable A. Finally, we exploit the spectral representation (5) to obtain

$$\langle A(x(t+\tau))A(x(t))\rangle = \sum_{i=0} \nu_i^\tau \int A(x)\phi_i(x)\mu(\mathrm{d}x) \int A(y)\psi_i(y)\,\mathrm{d}y. \tag{10}$$

For $i=0$ the RHS of (10) gives $\langle A\rangle^2$ so that

$$C_A(\tau) = \sum_{i=1} c_i(\nu_i)^\tau \tag{11}$$

and we recover (2) with the identification $\nu_k = \mathrm{e}^{-\alpha_k+\mathrm{i}\omega_k}$. The real part of $-\log\nu_k$ is therefore the mixing rate and its imaginary part provides the frequency of the corresponding oscillation (note that the resonances associated to ν_k are located at $\pm\mathrm{i}\log\nu_k$). Furthermore, if we assume that the eigenvalues $\{\nu_i\}$ have different moduli, we see from (10) and (11) that whenever the function A has non-zero projection on the eigenfunction $\psi_1(x)$, the asymptotic behaviour of the sum (11) is dominated by the term ν_1^τ (notice that if $1>|\nu_1|>|\nu_2|\rangle\ldots$, then the resonance $\mathrm{i}\log\nu_1$ is the one nearest to the real axis). For this reason, even in simple cases it is not easy to extract the next-to-nearest resonances $\mathrm{i}\log\nu_2$, $\mathrm{i}\log\nu_3\ldots$, as they control exponentially damped corrections to the asymptotic behaviour of the numerical signal.

In order to overcome this difficulty, we shall develop a numerical technique which, in a sense, is analogous to the method introduced by Benettin *et al* (1980), for the determination of Lyapunov spectra. In that case, one looks at the growth rate of a p-dimensional volume obtained by the external product of p different random vectors. Such an exponential rate is given by the sum of the first p Lyapunov exponents. In a similar way, we look at the time correlation functions of exterior forms of order p, obtained by the external product of p arbitrary functions. In the appendix, we shall show that their leading decay rate is related to the sum of the logarithms of the first p eigenvalues of the transfer operator. Operationally, we consider a set of p linearly independent functions $A_k(x)$ $(k=1,2,\ldots,p)$ and we compute each of them over p different trajectories

$$x_l(t)=f^t x_l(0) \qquad l=1,\ldots,p \tag{12}$$

where the $x_l(0)$'s are random initial conditions. If just one time series is available one could simply use p time shifted versions of this trajectory, provided that the shifts are larger than the correlation time.

Then, a p-form $\omega^{(p)}$ can be constructed from the determinant of the $p\times p$ matrix $\mathbf{O}$ whose elements are given by

$$(\mathbf{O})_{kl}=A_k(x_l). \tag{13}$$

A convenient choice for the above set of functions is $A_k(x)=x^{k-1}$, which yields

$$\omega^{(p)}=\det\mathbf{O}=\prod_{l>l'}(x_l-x_{l'}). \tag{14}$$

For instance, for $p=2$, we get

$$\omega^{(2)}=\det\begin{pmatrix}1 & x_1\\ 1 & x_2\end{pmatrix}. \tag{15}$$

Since $x_1(t)$ and $x_2(t)$ are independent trajectories we have $\langle x_1x_2\rangle=\langle x\rangle^2$, and

$$\tfrac{1}{2}\langle\omega^{(2)}(t+\tau)\omega^{(2)}(t)\rangle=\langle x(t+\tau)x(t)\rangle-\langle x\rangle^2. \tag{16}$$

The average in the LHS of (16) is therefore the correlation function of x. According to the previous discussion, its behaviour for large τ is dominated by ν_1 $(=\nu_0\cdot\nu_1)$. In a similar way (see the appendix), the asymptotic decay of a p-form correlation function is dominated by the product of the first p eigenvalues:

$$C^{(p)}(\tau)\equiv\frac{1}{p!}\langle\omega^{(p)}(t+\tau)\omega^{(p)}(t)\rangle\sim(\nu_0\nu_1\ldots\nu_{p-1})^\tau\qquad\text{for large }\tau \tag{17}$$

(notice that from (14) one gets $\langle\omega^{(p)}\rangle=0$ for any $p>1$).

A preliminary test of this method is obtained from the analysis of the piecewise linear expanding map $x(t+1)=x(t)/\theta$ for $0\leqslant x(t)\leqslant\theta$ and $x(t+1)=(x(t)-\theta)/(1-\theta)$ for $\theta<x(t)\leqslant 1$, of the interval [0, 1] onto itself (the Bernoulli map), for which the spectrum of the transfer operator is real and positive. The eigenvalues are (Mori *et al* 1981):

$$\nu_i=\theta^{i+1}+(1-\theta)^{i+1}. \tag{18}$$

Figure 1 shows the correlation decay of the p-forms (17) for $p=2, 3, 4, 5$, and $\theta=0.9$. The extracted mixing rates are in a very good agreement with the theoretical prediction $\Pi_{i=0}^{p-1}\nu_i$.

More interesting examples can be obtained by studying systems which exhibit intermittent behaviour, for example when the bifurcation parameter is close to a value where an attracting periodic orbit settles down. In this case, the real part of the nearest resonance can be straightforwardly associated with the period of the cycle which is about to become stable, whereas its imaginary part can be related to (the inverse of) the mean time the chaotic trajectory spends in the neighbourhood of such a cycle. We have analysed two situations of this type:

(1) the logistic map $x\to\alpha x(1-x)$ for $\alpha=1+\sqrt{8}-\varepsilon$, with $\varepsilon=10^{-4}$, that is very close to the stable window of period 3 (see Collet and Eckmann 1980);

(2) the Hénon map $x(t+1)=y(t)+1-ax(t)^2$, $y(t+1)=bx(t)$ of the plane, with the choice $a=1.31$, $b=0.3$ (we recall that for $a=1.3$, $b=0.3$ there exist an attracting periodic orbit of period 7).

In both these two examples we do not know what the spectrum of the operator $\mathscr{L}$ is, and therefore we shall speak about (complex) exponents (as they are defined in (2)), rather than eigenvalues. In order to extract such exponents from the numerical

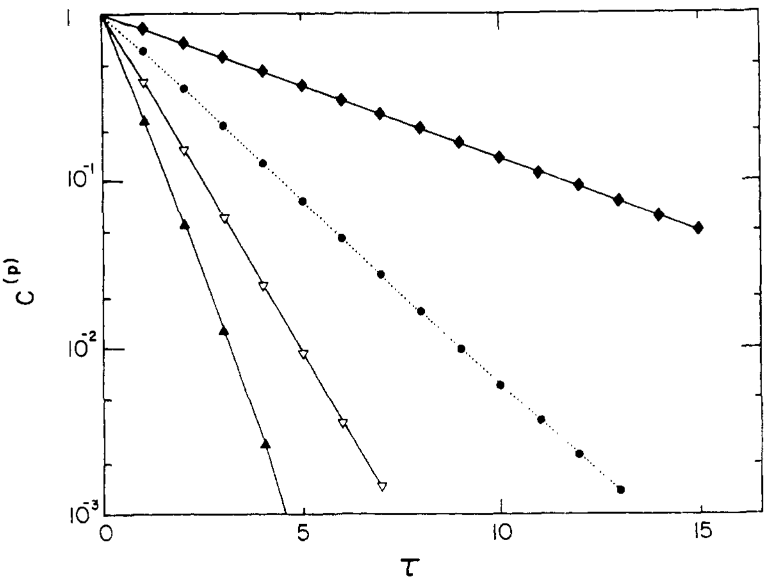

Figure 1. $C^{(p)}(\tau)$ plotted against τ, with $p = 2, 3, 4, 5$, for the Bernoulli map with $\theta = 0.9$, obtained by a numerical integration of 2.5×10^{-7} steps. The slopes of the linear scaling exhibited by $\ln C^{(p)}(\tau)$ are respectively 0.8201 ± 0.0005, 0.599 ± 0.001, 0.393 ± 0.002, 0.232 ± 0.003, to be compared with the theoretical values $\Pi_i^{p-1}\nu_i$, see (18).

signal we have adopted both a crude fitting procedure and a more refined technique based on Padé approximants (Isola 1988).

In the logistic map, with α as above, the behaviour of (standard) correlations is determined by the pair of complex conjugate exponents whose numerical values are $\simeq -0.017\pm i2\pi/3$ (the Lyapunov exponent is $\lambda = 0.208$). The 2-form correlation function therefore exhibits an oscillation of period three, as expected. Figure 2 shows, however, that this oscillation disappears using the 3-form, for which the leading decay rate is given by the sum of the above exponents $\simeq -0.034$. In addition, this result suggests a more appropriate indicator for the extraction of the decay rate, when modulated correlations (coming from the presence of a complex pair) are observed.

The Hénon map exhibits another interesting feature, illustrated in figure 3. The behaviour of the p-form correlation functions $C^{(p)}(\tau)$ for $p = 2, 3, 4, 5$ is well described by periodic oscillations with a damping factor given by $\exp[-0.020(p-1)\tau]$. Note that the maximum Lyapunov exponent is $\lambda = 0.247$. More precisely, the Fourier transform of $C^{(2)}$ exhibits peaks at the frequencies $\omega_k = k\times 2\pi/7$, with $k = 1, 2, \ldots 6$. Furthermore, the functions $C^{(p)}$, with $p \geqslant 3$, oscillate with frequencies given by combinations of the ω_k's (i.e. $k\times2\pi/7$ where $k = 0, 1, \ldots 6$). This indicates that at least the first six of the oscillations are not present, for $p \leqslant 6$.

Using the cycle expansion procedure (Artuso *et al* 1990) in the Hénon map, a preliminary calculation of the zeros of the Fredholm determinant $d(z) = \det(1 - z\mathscr{L})$ in the complex plane (Christiansen *et al* 1990) also suggests that many eigenvalues of $\mathscr{L}$ might have degenerate modulus. However, since the Hénon map is not hyperbolic, the cycle expansion is very slowly convergent. We shall report elsewhere a detailed study of the location of resonances for the Hénon map, at varying the control parameters. At present, a direct analysis of time correlations of exterior forms for non-axiom A systems can be very important to support these types of results.

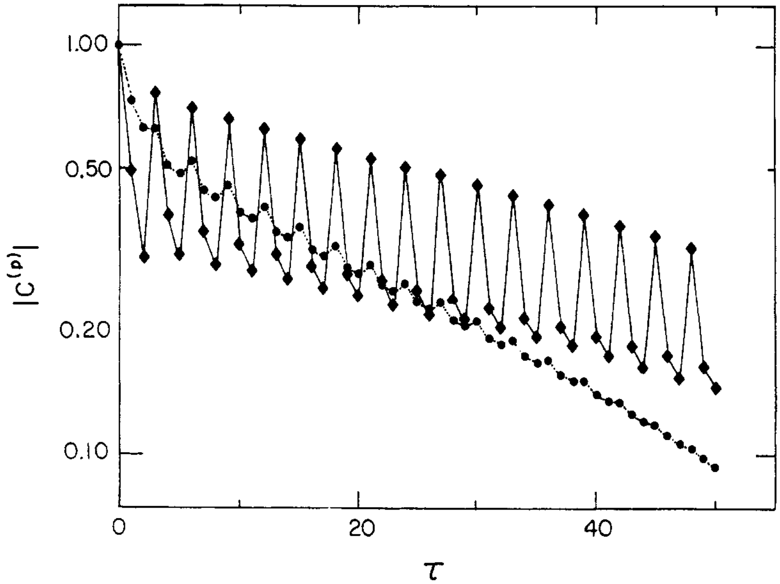

Figure 2. Absolute value of the correlation function of the exterior forms of order $p = 2$ (broken curve) and $p = 3$ (full curve) plotted against τ, for the logistic map with $\alpha = 3.82832\ldots$.

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Appendix

The time average of a function A of p independent trajectories $x_1, x_2, \ldots x_p$ is given by an integral over a probability measure obtained as the direct product of physical measures,

$$\langle A(x_1(t), \ldots x_p(t))\rangle = \prod_{n=1}^{p} \int \mu(\mathrm{d}x_n) A(x_1, \ldots, x_p). \tag{A1}$$

Assuming $\mu(\mathrm{d}x) = \psi_0(x)\,\mathrm{d}x$, it is useful to introduce a $p \times p$ diagonal matrix $(\mathbf{R})_{ij}(x) \equiv \psi_0(x_i)\delta_{ij}$ such that

$$\prod_{n=1}^{p} \int \mu(\mathrm{d}x_n) = \prod_{n=1}^{p} \int \mathrm{d}x_n \det[\mathbf{R}(x)]. \tag{A2}$$

Then, if $\mathbf{O}(x)$ denotes the $p \times p$ matrix $[\mathbf{O}(x)]_{jl} = A_j(x_l)$, we can write the average in (17) as

$$\langle \omega^{(p)}(t+\tau)\omega^{(p)}(t)\rangle = \prod_{n=1}^{p} \int \mathrm{d}x_n \det[(\mathbf{O} \circ f^{\tau})(x)] \det[\mathbf{O}(x)] \det[\mathbf{R}(x)]. \tag{A3}$$

Using the properties of the delta function and recalling that the product of determinants

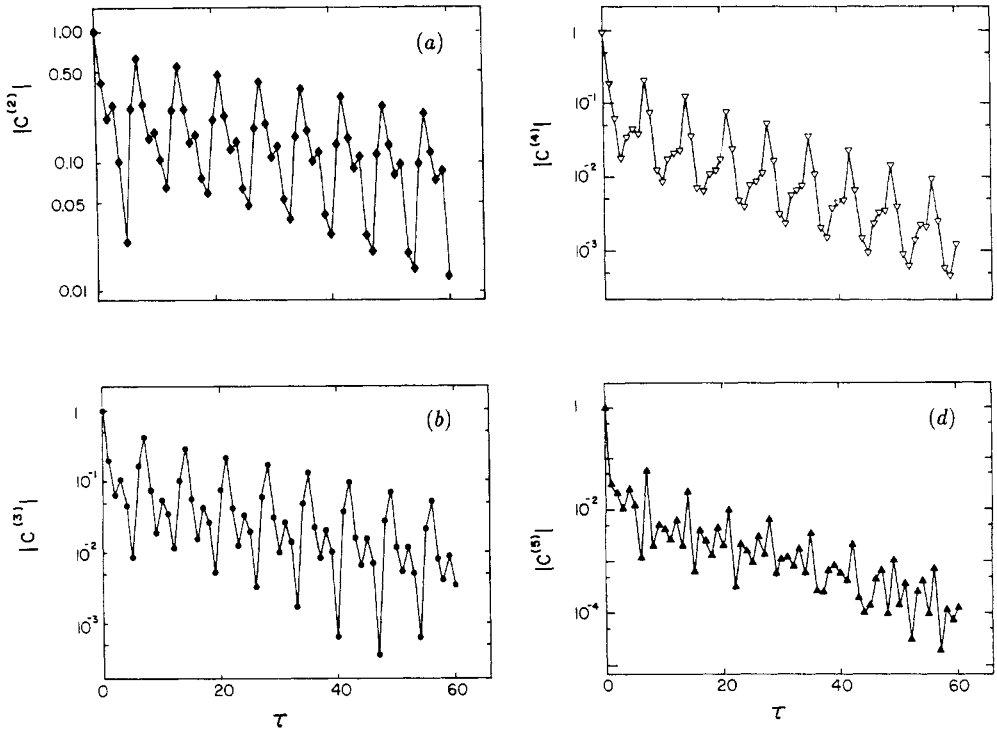

Figure 3. Absolute value of the correlation function of the exterior forms of order $p=2$ (figure 3(*a*)), $p=3$ (figure 3(*b*)), $p=4$ (figure 3(*c*)), and $p=5$ (figure 3(*d*)), for the Hénon map with $\alpha=1.31$, $\beta=0.3$, obtained by a numerical integration of 4×10^7 steps. The slopes extracted from the linear scaling exhibited by $\ln|C^{(p)}|(\tau)$ for τ large enough are respectively 0.020 ± 0.001, 0.041 ± 0.001, 0.060 ± 0.002, and 0.080 ± 0.003.

is the determinant of the product, (A3) becomes

$$\langle\omega^{(p)}(t+\tau)\omega^{(p)}(t)$$

$$=\prod_{n=1}^{p}\int \mathrm{d}x_n \prod_{l=1}^{p}\int \mathrm{d}y_l \det[\mathbf{O}(y)]\delta(y_l-f^\tau x_l)\det[\mathbf{OR}(x)]$$

$$=\prod_{l=1}^{p}\int \mathrm{d}y_l \det[\mathbf{O}(y)]\det[\mathbf{P}(y)] \tag{A4}$$

where we have introduced the $p\times p$ matrix $\mathbf{P}$ obtained by integrating over the variable x:

$$(\mathbf{P})_{lj}=\int \delta(y_l-f^\tau x)A_j(x)\psi_0(x)\,\mathrm{d}x=[\mathscr{L}^\tau\psi_0A_j](y_l). \tag{A5}$$

Inserting (A5) into (A4), we get

$$\langle\omega^{(p)}(t+\tau)\omega^{(p)}(t)\rangle=\prod_{l=1}^{p}\int \mathrm{d}y_l \det[\mathbf{OP}(y)]=p!\det\mathbf{Q} \tag{A6}$$

where

$$[\mathbf{OP}(y)]_{ij}=\sum_l O_{il}P_{lj}=\sum_l A_i(y_l)[\mathscr{L}^\tau\psi_0A_j](y_l) \tag{A7}$$

and the factor $p!$ comes from a permutation of the y_l's. Making use of the notation $(F|G)=\int \mathrm{d}y\, F(y)G(y)$ and of the spectral representation (10), one can verify that the $p\times p$ matrix $\mathbf{Q}$ in (A6) is given by

$$(\mathbf{Q})_{ij}=(A_i|\mathscr{L}^{\tau}\psi_0 A_j)=\sum_{r=0}^{\infty}\nu_r^{\tau}(A_i|\psi_r)(\phi_r|\psi_0 A_j). \tag{A8}$$

Now, the determinant of $\mathbf{Q}$ can be written as

$$\det[\mathbf{Q}]=\sum_{i_1=1}^{p}\dots\sum_{i_p=1}^{p}\varepsilon_{i_1\dots i_p}Q_{i_1 1}Q_{i_2 2}\dots Q_{i_p p}$$

by means of the Levi-Civita tensor $\varepsilon_{i_1\dots i_p}$. Putting everything together, the correlation function of a p-form is given by the expression

$$C^{(p)}(\tau)=\sum_{i_1=1}^{p}\dots\sum_{i_p=1}^{p}\varepsilon_{i_1\dots i_p}\sum_{r_1=0}^{\infty}\dots\sum_{r_p=0}^{\infty}(\nu_{r_1}\dots\nu_{r_p})^{\tau}$$
$$\times(\phi_{r_1}|\psi_0 A_1)\dots(\phi_{r_p}|\psi_0 A_p)(A_{i_1}|\psi_{r_1})\dots(A_{i_p}|\psi_{r_p}). \tag{A9}$$

On the other hand, it is easy to verify that

$$\sum_{i_1=1}^{p}\dots\sum_{i_p=1}^{p}\varepsilon_{i_1\dots i_p}(A_{i_1}|\psi_{r_1})\dots(A_{i_p}|\psi_{r_p}) \tag{A10}$$

is identically zero unless $r_1, r_2, \dots, r_p$ are distinct, e.g. regard (A10) as the determinant of the matrix $(\mathbf{M})_{ij}=\int A_i(y)\psi_{r_j}(y)\,\mathrm{d}y$. This means that, for $\tau\to\infty$, the leading term in (A9) is given by $(\nu_0\nu_1\dots\nu_{p-1})^{\tau}$ and the estimate (17) follows.

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